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Generalized q-oscillators and their Hopf structures

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Abstract. We study the relationships among the various forms of the q-oscillator algebra and consider the conditions under which it supports a Hopf structure. We also present a generalization of this algebra together with its corresponding Hopf structure. Its multimode extensions are also considered.

1. Introduction

Quantum groups, or more precisely the quantized universal enveloping algebras $\mathcal{U}_q(\mathcal{L})$ of Lie algebras $\mathcal{L}$, first emerged as the basic algebraic structures in the study of the quantum Yang–Baxter equations [1]. It was later shown by Drinfeld [2] that these structures could be described by a general class of associative algebras, called Hopf algebras, which are neither commutative nor co-commutative. Essentially, the non-co-commutativity is achieved by introducing a free parameter q which is usually called the deformation parameter.

One of the most well studied examples is that of the quantum group $\mathcal{U}_q(\mathrm{su}(2))$ (sometimes denoted as $\mathrm{su}_q(2)$) which was first considered by Skylanin [3] and independently by Kulish and Reshetikhin [4]. Recently, this algebra has been realized in terms of a q-analogue of the bosonic creation and annihiliation operators [5, 6]. Indeed, Macfarlane [5] introduced these q-oscillators a, a^+ by considering their action on a Hilbert space with basis $\{|n\rangle\}$, $n = 0, 1, 2, \ldots$ defined by

$$a|0\rangle = 0 \qquad |n\rangle = ([n]!)^{-1/2}(a^+)^n|0\rangle \tag{1}$$

where

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad \text{and} \qquad [n]! = [n][n-1][n-2]\ldots[1].$$

Then, by setting

$$a^+a = [N] \tag{2a}$$

$$aa^+ = [N+1] \tag{2b}$$

where N satisfies

$$N|n\rangle = n|n\rangle \tag{3a}$$

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he was able to furnish a representation for the q-oscillators

$$a^+|n\rangle = [n+1]^{1/2}|n+1\rangle \tag{3b}$$

$$a|n\rangle = [n]^{1/2}|n-1\rangle. \tag{3c}$$

Moreover, in this representation, one also has the following relations:

$$aa^+ - qa^+a = q^{-N} \tag{4}$$

$$aa^+ - q^{-1}a^+a = q^N \tag{5}$$

besides

$$[N, a^+] = a^+ \qquad [N, a] = -a. \tag{6}$$

Biedenharn [6] also independently arrived at similar results but instead of starting with relations (2), he postulated (4) and (6) with q replaced by $q^{1/2}$.

By using the Jordan–Schwinger construction, they gave a bosonic realization of $\mathrm{su}_q(2)$. Conversely, the q-oscillators can also be obtained directly from the usual representation of $\mathrm{su}_q(2)$. Ng [7] showed that by setting $j \to \infty$, $m \to \infty$ in the basis vectors spanning the Hilbert space of $\mathrm{su}_q(2)$, the q-oscillators can be obtained which satisfy relations (2).

Although the q-oscillators have been primarily used in giving realizations of quantum groups, they themselves may support a quantum group structure. Indeed, Hong Yan [8] showed that the q-oscillator algebra, when expressed in a symmetric form, could be endowed with a non-co-commutative Hopf structure. Instead of relations (2), he considered the commutator†

$$[a, a^+] = [N+1] - [N]. \tag{7}$$

Then, together with relations (6), he was able to construct a non-trivial Hopf algebra. It is worth noting that while relations (2) imply relation (7), the converse is not true; it is only in the representation (3) that the two are equivalent. In fact, the same holds for relations (4) and (5) with regard to (2) or (7).

In the following section, we discuss some issues pertaining to this inequivalence. In particular, we study the relationships among the various forms of the q-oscillators. Moreover, we also clarify some misleading notions in the literature about the q-oscillator algebra when regarded as a quantum group. In section 3, we present a generalized deformed-oscillator algebra which also has a Hopf structure. Here, Hong Yan's algebra is recoverd as a special case. The representation of this generalized algebra is also furnished. In section 4, we consider its multimode extensions. Besides a set consisting of mutually-commuting oscillators, we also present a multidimensional quantum group based on our generalization.

† In [8], the right-hand side is expressed as $[N+\frac{1}{2}] - [N-\frac{1}{2}]$. Here we have made the replacement $N \to N + \frac{1}{2}$ to be consistent with the notation that we have adopted in this paper.

2. q-oscillator algebras

The q-oscillator algebra consists of three elements a, a^+ and N defined by (6) together with *one* of the relations (2), (4), (5) or (7). In the following, we will examine how the various forms, namely, (2), (4), (5) and (7) of the oscillator algebra are related to each other. Here, relations (6) will be implicitly assumed as part of the algebra. For clarity, we consider two at a time.

Case (i): between (2) and (4). Starting with (2), we show that it implies (4). Indeed, by substituting (2) into the left-hand side of (4), we have

$$aa^+ - qa^+a = [N+1] - q[N] = q^{-N} \tag{8}$$

which is precisely the right-hand side of (4). On the other hand, to see whether (4) implies (2), we construct the Casimir operator for the algebra defined by (4) and (6). Now, it is easy to verify that

$$C_{(4)} = q^{-N}([N] - a^+a) \tag{9a}$$

commutes with all the operators, i.e. a, a^+, N. Thus, one can write

$$a^+a = [N] - q^N C_{(4)}. \tag{9b}$$

Moreover, we also have, using (4),

$$aa^+ = qa^+a + q^{-N} = [N+1] - q^{N+1}C_{(4)}. \tag{9c}$$

It is apparent then that (4) implies (2) only if $C_{(4)} = 0$. To show inequivalence, it is sufficient to show that there exist representations of (4) in which $C_{(4)}$ is non-zero. To this end, we have for $n = 0, 1, 2\ldots$ (see [9])

$$a^+|n\rangle = q^{-\nu_0/2}[n+1]^{1/2}|n+1\rangle \tag{10a}$$

$$a|n\rangle = q^{-\nu_0/2}[n]^{1/2}|n-1\rangle \tag{10b}$$

$$N|n\rangle = (\nu_0 + n)|n\rangle \tag{10c}$$

which suitably represents (4) and (6). Note that this representation carries a free parameter ν_0 and is more general than (3) above. In this representation, one has

$$C_{(4)}|n\rangle = q^{-\nu_0}[\nu_0]|n\rangle \tag{11}$$

which shows that for $\nu_0 \neq 0$, $C_{(4)}$ cannot be regarded as the null operator. Thus, we can surmise that (4), in general, does not imply (2).

Case (ii): between (2) and (5). Using arguments parallelling those above, it is easy to show that (2) implies (5) but not the converse. Here $C_{(5)}$ is similar to $C_{(4)}$ with $q \leftrightarrow q^{-1}$.

Case (iii): between (2) and (7). It is obvious that (2) implies (7). For the converse, we construct the Casimir operator $\mathcal{C}_{(7)}$ for (7) which reads as

$$\mathcal{C}_{(7)} = [N] - a^{+}a \tag{12a}$$

or

$$a^{+}a = [N] - \mathcal{C}_{(7)}. \tag{12b}$$

Using (7), we also have

$$aa^{+} = [N+1] - \mathcal{C}_{(7)}. \tag{12c}$$

From these, it is clear that (7) implies (2) only if $\mathcal{C}_{(7)} = 0$. Again for non-equivalence, it suffices to show that $\mathcal{C}_{(7)}$ is non-zero in some representation. In this case, one can construct the following representation:

$$a^{+}|n\rangle = ([n+1-\nu_0] + [\nu_0])^{1/2}|n+1\rangle \tag{13a}$$

$$a|n\rangle = ([n-\nu_0] + [\nu_0])^{1/2}|n-1\rangle \tag{13b}$$

$$N|n\rangle = (n-\nu_0)|n\rangle \tag{13c}$$

for $n = 0, 1, 2\ldots$ in which

$$\mathcal{C}_{(7)}|n\rangle = -[\nu_0]|n\rangle. \tag{14}$$

It is evident then that $\mathcal{C}_{(7)} \neq 0$ for $\nu_0 \neq 0$.

Case(iv): between (4) and (5). From (9*b*) and (9*c*) we have

$$\begin{aligned} aa^{+} - q^{-1}a^{+}a &= [N+1] - q^{N+1}\mathcal{C}_{(4)} - q^{-1}([N] - q^{N}\mathcal{C}_{(4)}) \\ &= q^{N} - (q-q^{-1})q^{N}\mathcal{C}_{(4)} \end{aligned} \tag{15}$$

which shows that (4) is not equivalent to (5) since, in general, $\mathcal{C}_{(4)} \neq 0$. Similarly, (5) does not imply (4).

Case (v): between (4) and (7). From (9*b*) and (9*c*), we obtain

$$aa^{+} - a^{+}a = [N+1] - [N] - (q-1)q^{N}\mathcal{C}_{(4)} \tag{16}$$

which means that (4) does not imply (7). Conversely, from (12*b*) and (12*c*), we have

$$\begin{aligned} aa^{+} - qa^{+}a &= [N+1] - q[N] - \mathcal{C}_{(7)} + q\mathcal{C}_{(7)} \\ &= q^{-N} + (q-1)\mathcal{C}_{(7)} \end{aligned} \tag{17}$$

which again demonstrates the inequivalence.

Case (vi): Between (5) and (7). Arguments and conclusions similar to case (v) with $q \leftrightarrow q^{-1}$.

From the results above we can surmise that although the various forms are interchangeable in the representation (3), they are nevertheless inequivalent at the algebraic level. This distinction becomes particularly important when one deals with pure algebraic constructs. For instance, when we are considering the Hopf structure of the q-oscillators, it is necessary to distinguish the relation in question. Now, due to the equivalence of (2), (4), (5) and (7) *in the representation* (3), it is sometimes implied that they are all Hopf algebras. Indeed, it has been claimed (see [10]) that relations (4) *together with* (5) which imply (2) and, hence, (7) have a Hopf structure defined by†

$$\Delta(a^+) = (a^+ \otimes q^{1/2(N+1/2)} + \mathrm{i}q^{-1/2(N+1/2)} \otimes a^+)\mathrm{e}^{-\mathrm{i}\theta/2} \tag{18a}$$

$$\Delta(a) = (a \otimes q^{1/2(N+1/2)} + \mathrm{i}q^{-1/2(N+1/2)} \otimes a)\mathrm{e}^{-\mathrm{i}\theta/2} \tag{18b}$$

$$\Delta(N) = N \otimes \mathbf{1} + \mathbf{1} \otimes N + \gamma \mathbf{1} \otimes \mathbf{1} \tag{18c}$$

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \tag{18d}$$

$$\epsilon(a^+) = \epsilon(a) = 0 \tag{18e}$$

$$\epsilon(N) = -\gamma \qquad \epsilon(\mathbf{1}) = 1 \tag{18f}$$

$$S(a^+) = -q^{1/2}a^+ \qquad S(a) = -q^{-1/2}a \tag{18g}$$

$$S(N) = -N - 2\gamma\mathbf{1} \qquad S(\mathbf{1}) = \mathbf{1} \tag{18h}$$

where $\gamma = \frac{1}{2} - \mathrm{i}\theta/\ln q$ and $\theta = (\pi/2) + 2\pi l$, $l \in \mathbb{Z}$. Here, the maps Δ, ϵ and S which are the coproduct, co-unit and antipode, respectively, satisfy

$$\Delta(ab) = \Delta(a)\Delta(b) \qquad \epsilon(ab) = \epsilon(a)\epsilon(b) \qquad S(ab) = S(b)S(a) \tag{19}$$

for any two elements a, b of the Hopf algebra. In other words, they are algebra homomorphisms (antihomomorphism for the case of S). Although these constitute a Hopf structure for relation (7) together with (6), they do not for relations (4), (5) and (2). To see why this is so, consider relation (4) as an example. By applying Δ to both sides, we have, using (19),

$$\begin{aligned}\Delta(a)\Delta(a^+) - q\Delta(a^+)\Delta(a) = -\mathrm{i}\{&q^{-N} \otimes q^{N+1/2} + \mathrm{i}(1-q)q^{-1/2}q^{-1/2N}a \otimes q^{1/2N}a^+ \\ &- q^{-(N+1/2)} \otimes q^{-N} + \mathrm{i}(1-q)q^{1/2}q^{-1/2N}a^+ \otimes q^{1/2N}a\}\end{aligned} \tag{20}$$

whereas

$$\Delta(q^{-N}) = \mathrm{i}q^{-1/2}q^{-N} \otimes q^{-N} \tag{21}$$

which means that

$$\Delta(a)\Delta(a^+) - q\Delta(a^+)\Delta(a) \neq \Delta(q^{-N}). \tag{22}$$

Similarly, one can easily verify that Δ is also incompatible with (2) and (5). Thus, the Hopf structure is valid only for relation (7) and not for the rest.

† In [10], $q^{1/2}$ is used instead of q.

3. Generalized q-oscillator algebra

Recently, some authors [11] have considered a generalized version of (4)

$$aa^+ - qa^+a = q^{\alpha N+\beta} \tag{23}$$

together with (6) and attempted to give it a Hopf structure. However, the coproduct defined there fails the compatibility requirement (5) and (2) in the same way as (18a) and (18b) failed for relations (4). Specifically, the proposed coproduct

$$\Delta(a^+) = a^+ \otimes q^{1/2(\alpha N+\beta)} + q^{-1/2(\alpha N+\beta)} \otimes a^+ \tag{24a}$$

$$\Delta(a) = a \otimes q^{1/2(\alpha N+\beta)} + q^{-1/2(\alpha N+\beta)} \otimes a \tag{24b}$$

$$\Delta(N) = N \otimes \mathbf{1} + \mathbf{1} \otimes N + (\beta/\alpha)\mathbf{1} \otimes \mathbf{1} \tag{24c}$$

fails with respect to (23)

$$\Delta(a)\Delta(a^+) - q\Delta(a^+)\Delta(a) \neq \Delta(q^{\alpha N+\beta}). \tag{24}$$

In this section, we furnish a generalized version of the q-oscillator algebra which can be endowed with a Hopf structure.

As seen from the previous section, among the various forms of the q-oscillator algebra, relation (7) is the only version that supports a Hopf structure. So, it is conceivable that any generalization would most likely be based on (7) rather than (4). To write down a generalized version of (7), it is instructive to consider again the relationship between (4) and (7). Instead of using a representation in which $C_{(4)}$ is zero (see (16)), one can also obtain (7) by considering *both* (4) *and* (5). When taken together, they imply (2) which in turn implies (7). It is worth noting that (5) is the $q \leftrightarrow q^{-1}$ analogue of (4). Now, let us apply this procedure to (23). Since α and β are arbitrary, we can replace $\alpha \to -\alpha$ and $\beta \to -\beta$ and rewrite (23) as

$$aa^+ - qa^+a = q^{-\alpha N-\beta}. \tag{26}$$

Its $q \leftrightarrow q^{-1}$ analogue is given by

$$aa^+ - q^{-1}a^+a = q^{\alpha N+\beta} \tag{27}$$

which together with (26) implies that

$$aa^+ = [\alpha N + \beta] \tag{28a}$$

$$a^+a = [\alpha N + \beta + 1]. \tag{28b}$$

This in turn leads us to

$$[a, a^+] = [\alpha N + \beta + 1] - [\alpha N + \beta] \tag{29}$$

or

$$[a, a^+] = [\alpha N + \beta_1] - [\alpha N + \beta_2] \tag{30a}$$

where

$$\beta_1 - \beta_2 = 1 \tag{30b}$$

which is the generalization of (7) we sought. To be as general as possible, we will also modify (6) somewhat by introducing a free parameter η into the commutation relations

$$[N, a^+] = \eta a^+ \qquad [N, a] = -\eta a. \tag{31}$$

It is worth noting that the algebra composed of (30) and (31) admits a non-trivial central term which is given by

$$C = a^+ a - \frac{1}{2} \frac{\sinh(\epsilon(\alpha N + \beta + \frac{1}{2} - \frac{1}{2}\alpha\eta))}{\cosh(\epsilon/2)\sinh(\epsilon\alpha\eta/2)}. \tag{32}$$

Now, it is also important to note that, as in the case between (2) and (7), relations (30) do not necessarily imply (28) although the latter have been used in constructing the former. This can be demonstrated easily by constructing a representation of (30) and (31) in which (26), (27) and (28) do not hold individually. Indeed, in the basis $\{|n\rangle\}$, $n = 0, 1, 2, \ldots$, a representation of (30) and (31) is given by

$$a|n\rangle = \left\{ \frac{\cosh(\epsilon\alpha(\nu_0 + (n-1)\eta/2) + \epsilon(\beta + \frac{1}{2}))\sinh(\epsilon\eta\alpha n/2)}{\cosh(\epsilon/2)\sinh(\epsilon\eta\alpha/2)} \right\}^{1/2} |n-1\rangle \tag{33a}$$

$$a^+|n\rangle = \left\{ \frac{\cosh(\epsilon\alpha(\nu_0 + n\eta/2) + \epsilon(\beta + \frac{1}{2}))\sinh(\epsilon\eta\alpha(n+1)/2)}{\cosh(\epsilon/2)\sinh(\epsilon\eta\alpha/2)} \right\}^{1/2} |n+1\rangle \tag{33b}$$

$$N|n\rangle = (\nu_0 + n\eta)|n\rangle \tag{33c}$$

where we have taken $\epsilon = \ln q$ and ν_0 is a free parameter which characterizes the representation. In this representation, it is not difficult to see the inequivalence between (30) and any one of (26)–(28). This is true even when η is set to 1.

Now, let us turn to the Hopf structure associated with (30) and (31). We start by considering the associative algebra $\mathcal{H}$ generated by $\{\mathbf{1}, a^+, a, N\}$ and postulating the following for the coproduct, co-unit and antipode:

$$\Delta(a^+) = c_1 a^+ \otimes q^{\alpha_1 N} + c_2 q^{\alpha_2 N} \otimes a^+ \tag{34a}$$

$$\Delta(a) = c_3 a \otimes q^{\alpha_3 N} + c_4 q^{\alpha_4 N} \otimes a \tag{34b}$$

$$\Delta(N) = c_5 N \otimes \mathbf{1} + c_6 \mathbf{1} \otimes N + \gamma \mathbf{1} \otimes \mathbf{1} \tag{34c}$$

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \tag{34d}$$

$$\epsilon(a^+) = c_7 \qquad \epsilon(a) = c_8 \tag{34e}$$

$$\epsilon(N) = c_9 \qquad \epsilon(\mathbf{1}) = 1 \tag{34f}$$

$$S(a^+) = -c_{10} a^+ \qquad S(a) = -c_{11} a \tag{34g}$$

$$S(N) = -c_{12} N + c_{13} \mathbf{1} \qquad S(\mathbf{1}) = \mathbf{1}. \tag{34h}$$

Here, c_i ($i = 1, 2, \ldots 13$), α_i ($i = 1, 2, \ldots 4$) and γ are constants to be determined. These constants are obtained by requiring that Δ, ϵ and S satisfy the co-associativity, co-unit and antipode axioms, respectively,

$$(\mathrm{id} \otimes \Delta)\Delta(h) = (\Delta \otimes \mathrm{id})\Delta(h) \tag{35a}$$

$$(\mathrm{id} \otimes \epsilon)\Delta(h) = (\epsilon \otimes \mathrm{id})\Delta(h) = h \tag{35b}$$

$$m(\mathrm{id} \otimes S)\Delta(h) = m(S \otimes \mathrm{id})\Delta(h) = \epsilon(h)\mathbf{1} \tag{35c}$$

where $h \in \mathcal{H}$ and $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ is the multiplication map. By substituting the different generators of $\mathcal{H}$ into (35) and noting that

$$a^+ q^{\alpha N} = q^{-\alpha\eta} q^{\alpha N} a^+ \tag{36a}$$

$$a q^{\alpha N} = q^{\alpha\eta} q^{\alpha N} a \tag{36b}$$

for an arbitrary α, we obtain

$$\begin{array}{llllll} c_1 = q^{\alpha_1\gamma} & c_2 = q^{\alpha_2\gamma} & c_3 = q^{\alpha_3\gamma} & c_4 = q^{\alpha_4\gamma} & c_5 = 1 & c_6 = 1 \\ c_7 = 0 & c_8 = 0 & c_9 = -\gamma & c_{10} = q^{\alpha_1\eta} & c_{11} = q^{-\alpha_3\eta} & c_{12} = 1 \\ c_{13} = -2\gamma & \alpha_2 = -\alpha_1 & \alpha_4 = -\alpha_3 & & & \end{array} \tag{37}$$

which essentially fixes 15 of the 18 constants in (34). We must also require that Δ, ϵ and S be algebra homomorphisms. Here, further constraints arise when we set

$$\Delta(a)\Delta(a^+) - \Delta(a^+)\Delta(a) = \Delta([\alpha N + \beta_1] - [\alpha N + \beta_2]). \tag{38}$$

For this to be satisfied, we must impose the following:

$$q^{(\alpha_1 - \alpha_3)\eta} = 1 \tag{39a}$$

$$\alpha_1 + \alpha_3 = \alpha \tag{39b}$$

$$q^{2\alpha\gamma} = -q^{\beta_1 + \beta_2}. \tag{39c}$$

With these, the homomorphisms ϵ and S entail no further constraints. For real q, equations (39) imply that

$$\alpha_1 = \alpha_3 = \tfrac{1}{2}\alpha \tag{40a}$$

and

$$\gamma = \frac{\beta_1 + \beta_2}{2\alpha} - \frac{\mathrm{i}(2k+1)\pi}{2\alpha \ln q} \qquad k \in \mathbb{Z} \tag{40b}$$

which now fixes all the constants in (34).

To summarize briefly, the Hopf structure for $\mathcal{H}$, with defining relations (30) and (31), reads as

$$\Delta(a^+) = a^+ \otimes q^{\alpha(N+\gamma)/2} + q^{-\alpha(N+\gamma)/2} \otimes a^+ \tag{41a}$$

$$\Delta(a) = a \otimes q^{\alpha(N+\gamma)/2} + q^{-\alpha(N+\gamma)/2} \otimes a \tag{41b}$$

$$\Delta(N) = N \otimes \mathbf{1} + \mathbf{1} \otimes N + \gamma \mathbf{1} \otimes \mathbf{1} \tag{41c}$$

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \tag{41d}$$

$$\epsilon(a^+) = \epsilon(a) = 0 \tag{41e}$$

$$\epsilon(N) = -\gamma \qquad \epsilon(\mathbf{1}) = 1 \tag{41f}$$

$$S(a^+) = -q^{\alpha\eta/2} a^+ \qquad S(a) = -q^{-\alpha\eta/2} a \tag{41g}$$

$$S(N) = -N - 2\gamma\mathbf{1} \qquad S(\mathbf{1}) = \mathbf{1} \tag{41h}$$

where γ satisfies (40*b*). Note that by setting $\eta = 1$, $\beta_1 = 1$, $\beta_2 = 0$, $\alpha = 1$ and putting $k = 2l$, $l \in \mathbb{Z}$ in (40*b*) we recover the Hopf structure associated with (6) and (7).

4. Multimode q-oscillators

Various multimode extensions of the q-oscillators have been proposed [4, 5, 12, 13]. In particular, the extension of (4) *and* (5) or equivalently that of (2) consist of taking p independent oscillators (mutually commuting) $\{a_i, a_i^+, N_i | i = 1, 2, \ldots p\}$ with the relations [13]

$$a_i a_j^+ - (1 + \delta_{ij}(q-1)) a_j^+ a_i = \delta_{ij} q^{-N_i} \tag{42a}$$

$$a_i a_j^+ - (1 + \delta_{ij}(q^{-1}-1)) a_j^+ a_i = \delta_{ij} q^{N_i} \tag{42b}$$

$$[a_i, a_j] = [a_i^+, a_j^+] = 0 \tag{42c}$$

$$[N_i, a_j] = -\delta_{ij} a_j \qquad [N_i, a_j^+] = \delta_{ij} a_j^+. \tag{42d}$$

Here, we present a multimode extension of (30) and show that it also supports a non-co-commutative Hopf structure. To this end, we propose the following relations for the set of p oscillators:

$$[a_i, a_j^+] = ([\alpha_i N_i + \beta_i + 1] - [\alpha_i N_i + \beta_i])\delta_{ij} \tag{43a}$$

$$[a_i, a_j] = [a_i^+, a_j^+] = 0 \tag{43b}$$

$$[N_i, a_j] = -\eta_i a_j \delta_{ij} \qquad [N_i, a_j^+] = \eta_i a_j^+ \delta_{ij} \tag{43c}$$

where α_i, β_i and η_i ($i = 1, 2, \dots p$) are free parameters. Then, it can easily be shown that the associative algebra generated by $\{\mathbf{1}, a_i, a_i^+, N_i\}$ ($i = 1, 2, \dots p$) with the above defining relations admits the following non-co-commutative Hopf structure:

$$\Delta(a_i^+) = a_i^+ \otimes q^{\alpha_i(N_i+\gamma_i)/2} + q^{-\alpha_i(N_i+\gamma_i)/2} \otimes a_i^+ \tag{44a}$$

$$\Delta(a_i) = a_i \otimes q^{\alpha_i(N_i+\gamma_i)/2} + q^{-\alpha_i(N_i+\gamma_i)/2} \otimes a_i \tag{44b}$$

$$\Delta(N_i) = N_i \otimes \mathbf{1} + \mathbf{1} \otimes N_i + \gamma_i \mathbf{1} \otimes \mathbf{1} \tag{44c}$$

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \tag{44d}$$

$$\epsilon(a_i^+) = \epsilon(a_i) = 0 \tag{44e}$$

$$\epsilon(N_i) = -\gamma_i \qquad \epsilon(\mathbf{1}) = 1 \tag{44f}$$

$$S(a_i^+) = -q^{\alpha_i \eta_i/2} a_i^+ \qquad S(a_i) = -q^{-\alpha_i \eta_i/2} a_i \tag{44g}$$

$$S(N_i) = -N_i - 2\gamma_i \mathbf{1} \qquad S(\mathbf{1}) = \mathbf{1} \tag{44h}$$

with

$$\gamma_i = \frac{2\beta_i + 1}{2\alpha_i} - \frac{\mathrm{i}(2k_i + 1)\pi}{2\alpha_i \ln q} \qquad k_i \in \mathbb{Z}. \tag{45}$$

In verifying the homomorphism property, we have used

$$a_i^+ q^{\rho N_j} = q^{-\rho\eta_j\delta_{ij}} q^{\rho N_j} a_i^+ \tag{46a}$$

$$a_i q^{\rho N_j} = q^{\rho\eta_j\delta_{ij}} q^{\rho N_j} a_i \tag{46b}$$

for an arbitrary ρ. Here, the Hopf structure is essentially the Hopf structure of each oscillator taken independently. It is interesting to note that the oscillators can also be coupled in a non-trivial way. To see how this can be accomplished, let us examine relations (39) closely. If we allow η to be complex and relate it to q via

$$\eta = \frac{2\pi\mathrm{i}}{\ln q} \tag{47}$$

then relation (39*a*) implies that $\alpha_1 - \alpha_3 = l$ ($l \in \mathbb{Z}$). Now, this means that we can assign integer values to α_1 and α_3 which in turn allows the indexing of oscillators. For instance, if we set $\alpha_1 = m$ and $\alpha_3 = n$ then the oscillators can be indexed as a_m and a_n^+, respectively. This effectively permits a number of oscillators to be considered together. Moreover, with α also being integer valued, as a consequence of (39*b*), the commutation relations between the various oscillators become non-trivial. As for (39*c*), we have

$$\beta_1 + \beta_2 = 2(\alpha_1 + \alpha_3)\gamma + \mathrm{i}\frac{(2k+1)\pi}{\ln q} \qquad k \in \mathbb{Z}. \tag{48}$$

Then, by putting $k = 0$ (for simplicity) and using (30*b*), we obtain

$$\beta_1 = (\alpha_1 + \alpha_3)\gamma + \frac{\mathrm{i}\pi}{2\ln q} + \frac{1}{2} \tag{49a}$$

$$\beta_2 = (\alpha_1 + \alpha_3)\gamma + \frac{\mathrm{i}\pi}{2\ln q} - \frac{1}{2}. \tag{49b}$$

With these, we have†

$$[a_m, a_n^+] = \left[(m+n)(N+\gamma) + \frac{\mathrm{i}\pi}{2\ln q} + \frac{1}{2}\right] - \left[(m+n)(N+\gamma) + \frac{\mathrm{i}\pi}{2\ln q} - \frac{1}{2}\right]$$
$$= \mathrm{i}\frac{\sinh(\epsilon(m+n)(N+\gamma))}{\cosh(\epsilon/2)} \tag{50}$$

where we have taken $q = e^{\epsilon}$. Thus, the commutation relations for a system of p oscillators can be written as

$$[a_m, a_n^+] = \mathrm{i}\frac{\sinh(\epsilon(m+n)(N+\gamma))}{\cosh(\epsilon/2)} \tag{51a}$$

$$[a_m, a_n] = [a_m^+, a_n^+] = 0 \tag{51b}$$

$$[N, a_m] = -\frac{2\pi\mathrm{i}}{\ln q}a_m \qquad [N, a_m^+] = \frac{2\pi\mathrm{i}}{\ln q}a_m^+ \tag{51c}$$

where $m, n = 1, 2, \ldots p$. It is important to note that unlike the previous case we have only one N operator. The corresponding Hopf structure is then given by

$$\Delta(a_m^+) = a_m^+ \otimes q^{m(N+\gamma)} + q^{-m(N+\gamma)} \otimes a_m^+ \tag{52a}$$

$$\Delta(a_m) = a_m \otimes q^{m(N+\gamma)} + q^{-m(N+\gamma)} \otimes a_m \tag{52b}$$

$$\Delta(N) = N \otimes \mathbf{1} + \mathbf{1} \otimes N + \gamma\mathbf{1} \otimes \mathbf{1} \tag{52c}$$

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \tag{52d}$$

$$\epsilon(a_m^+) = \epsilon(a_m) = 0 \tag{52e}$$

$$\epsilon(N) = -\gamma \qquad \epsilon(\mathbf{1}) = 1 \tag{52f}$$

$$S(a_m^+) = -a_m^+ \qquad S(a_m) = -a_m \tag{52g}$$

$$S(N) = -N - 2\gamma\mathbf{1} \qquad S(\mathbf{1}) = \mathbf{1}. \tag{52h}$$

5. Conclusion

In this paper, we have considered the various forms of the q-oscillator algebra and shown that, contrary to the commonly held notion, they are actually not equivalent. It is also pointed out that the Hopf structure found for one of these versions does not extend to the rest by virtue of this inequivalence. For the algebra that is a quantum group, we have given its generalization together with the associated Hopf structure. Based on this generalization, we have also furnished two multimode extensions. In the first example, we have considered a set of (mutually commuting) independent oscillators and shown that the Hopf structure of each oscillator system extends naturally to the multimode case. For the second example, we have presented a Hopf algebra comprising of a set of non-commuting oscillators.

† Here, we have set $\alpha_1 = m$ and $\alpha_3 = n$ and the corresponding oscillators by a_m and a_n^+, respectively.

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